Figure 1 indicates that the collective pitch should be dropped to its lowest setting (0 deg) immediately after T_D s. This action reduces the induced power loss, and effectively arrests the high rate of rotor rpm loss. The collective pitch is then kept at 0 deg for 1.5 s before a collective flare is executed to generate lift to reduce the impact velocity. The collective pitch is on (or near) its upper bound during the last 0.5 s of travel. From this particular entry condition (with $h_0 = 100$ ft), neither the upper nor lower bound on the rotor rpm is violated.

The upper bound on the rotor rpm is violated when T_D is 0.1 s (with standard rotor blades) and the entry height is above 450 ft. Figure 2 shows the time histories of the rotor rpm obtained with and without a Ω_{max}-bound from an entry height of 500 ft. Without the Ω_{min}-bound, the peak rotor rpm is 415 rpm. The rotor will peak at an even higher rpm if the engine fails at a higher entry height. With a Ω_{max}-bound, the rotor rpm stays on the bound for a brief period of time, but never exceeded it. The Ω_{max}-bound is not active in this particular case.

In Fig. 3, the impact velocity is plotted against the entry height for three rotor lock numbers of 5.43, 3.19, and 2.61. For all of the cases studied, the impact velocity first increases, reaches a maximum, and then decreases with the entry height. For the cases studied, the critical entry height, i.e., the entry height with the maximum impact velocity, is between 100 and 150 ft. This critical height is close to the height of the "knee" of the low-speed H-V restriction zone. Figure 3 also indicates that the high-inertia rotor (with a smaller lock number) performs better than the low-inertia rotor for all of the entry heights studied.

Figure 4, which depicted results obtained for three pilot reaction times of 2, 1, and 0.1 s, indicates that the longer the pilot reaction time, the poorer the autorotational performance. For example, from an entry height of 150 ft, the impact velocity obtained with a 2-s reaction time is more than 50% higher than that found with a 0.1-s reaction time. This deterioration is due mainly to the severe bleeding of the rotor rpm, loss of vehicle altitude, as well as the buildup of vertical sinkrate within the pilot reaction time.

Concluding Remarks

A point-mass model of an OH-58A helicopter was used to determine its autorotation profiles that minimize the impact velocity of the helicopter while staying within bounds on the rotor's collective pitch and angular speed. The optimal control strategies obtained are similar to those used by pilots in autorotational landings. The study indicates that there is a potential for reducing the H-V restriction zone of OH-58A helicopters using optimal energy management techniques.

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References

[1]Wood, T. L. "High Energy Rotor System," The 32nd Annual National Forum of the American Helicopter Society, Washington, DC, May 1976.

[2]Johnson, W., "Helicopter Optimal Descent and Landing after Power Loss," NASA TM 73,244, May 1977.

[3]Lee, A. Y., Bryson, A. E., Jr., and Hindson, W. S., "Optimal Landing of a Helicopter in Autorotation," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 1, 1988, pp. 7–12.

[4]Nelder, J. A., and Mead, R. A., "Simplex Method for Function Minimization," *Computer Journal*, Vol. 7, No. 4, 1965, pp. 308–313.

[5]Horng, I. R., and Chou, J. H., "Design of Optimal Observers with Specified Eigenvalues via Shifted Legendre Polynomials," *Journal of Optimization Theory and Applications,* Vol. 51, No. 1, 1986, pp. 179–188.

How to Perform Differentiations on Matrices

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Introduction

In many linear system studies and formulations, a derivative of a matrix equation with respect to a matrix is desirable. Unfortunately, there is no easy way of performing such differentiation. This Note introduces a method of differentiating matrices easily through the definition of some unique operators and notations.

Definitions

Definition: Vec (·) Operator for Two-Dimensional Arrays[1]

For a two-dimensional array A of dimension $p \times q$

$$\mathrm{Vec}(A) \equiv \left[a_1^T a_2^T \cdots a_q^T\right]^T$$

where a_i is the ith column of array A. Note that the Vec operator "vectorizes" a two-dimensional array by stacking the columns together and reducing the dimension of the operand array by 1, i.e., from two dimensions to one dimension.

Example:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\mathrm{Vec}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}^T$$

Definition: Vec (·) Operator for Three-Dimensional Arrays

For a three-dimensional array B of dimension (m,n,p)

$$\mathrm{Vec}(B) \equiv \left[\mathrm{Vec}(B_{IJ,1}),\ \mathrm{Vec}(B_{IJ,2}) \cdots \mathrm{Vec}(B_{IJ,p})\right]$$

where $B_{IJ,k}$ is the kth subarray of B, where $1 \le k \le p$, $I = 1,\ldots,m$, $J = 1,\ldots,n$. Note that the Vec operator reduces the dimension of the operand array by 1, i.e., from three dimensions to two dimensions.

Example:

$$B(2,2,3) = \left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 9 & 11 \\ 10 & 12 \end{pmatrix}\right]$$

$$\mathrm{Vec}(B) = \left[\mathrm{Vec}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix},\ \mathrm{Vec}\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix},\ \mathrm{Vec}\begin{pmatrix} 9 & 11 \\ 10 & 12 \end{pmatrix}\right]$$

$$= \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

Definition: Expanded Identity Matrix II

II is a three-dimensional array of dimension (x,y,xy), and Vec(II) is the identity matrix of dimension (xy,xy).

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Example: For $x=2$, $y=3$

$$\mathrm{II}(2,3,6)=\left[\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right]$$

$$\mathrm{Vec}\,(\mathrm{II})=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}=\mathrm{I}_{6\times 6}$$

Definition: Transposition of the Expanded Identity Matrix $\mathrm{II}=\mathrm{II}(x,y,xy)$ and $\mathrm{II}^T=\mathrm{II}^T(y,x,xy)$

Example: For $x=2$, $y=3$

$$\mathrm{II}^T=\left[\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}^T,\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}^T,\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}^T,\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}^T,\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}^T,\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}^T\right]$$

$$=\left[\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix}\right]$$

Definition: Expanded Identity Matrix with Respect to a Two-Dimensional Array

$\mathrm{II}_{(\cdot)}$ denotes an expanded identity matrix of dimensions defined by the operand array.

Example: $A=A(2,3)$ and $\mathrm{II}_A=\mathrm{II}(2,3,6)$.

Properties of the Vec Operator and the Expanded Identity Matrices

Differentiation: For a Two-Dimensional Array B of Dimension $m\times n$

$$\frac{\partial B}{\partial\,\mathrm{Vec}(B)}=\mathrm{II}_B$$

Proof: Consider the (i,j) element of $\mathrm{Vec}(\partial B/\partial\,\mathrm{Vec}(B))$, where $1\le i\le mn$ and $1\le j\le mn$.

$$\mathrm{Vec}\left(\frac{\partial B}{\partial\,\mathrm{Vec}(B)}\right)_{i,j}=\frac{\partial B_{p,q}}{\partial\,\mathrm{Vec}(B)_j}\qquad q<n,\quad pm+q=i$$

$$=\frac{\partial\,\mathrm{Vec}(B)_i}{\partial\,\mathrm{Vec}(B)_j}$$

$$=\left(\frac{\partial\,\mathrm{Vec}(B)}{\partial\,\mathrm{Vec}(B)}\right)_{i,j}$$

Therefore

$$\mathrm{Vec}\left(\frac{\partial B}{\partial\,\mathrm{Vec}(B)}\right)=\left(\frac{\partial\,\mathrm{Vec}(B)}{\partial\,\mathrm{Vec}(B)}\right)=\mathrm{I}_{mn\times mn}=\mathrm{Vec}(\mathrm{II}_B)$$

Also, since the dimension of $\partial B/\partial\,\mathrm{Vec}(B)$ is equal to the dimension of II_B, it follows that

$$\frac{\partial B}{\partial\,\mathrm{Vec}(B)}=\mathrm{II}_B$$

Note that the Vec $(\cdot)$ operator is not generally reversible. The dimension of the operand must be specified for the Vec $(\cdot)$ operator to be reversible. Similarly

$$\frac{\partial B^T}{\partial\,\mathrm{Vec}(B)}=\mathrm{II}_B^T$$

Chain Rule: The following differentiation chain rules can easily be verified by expanding the matrices and using the preceding basic differentiation rules:

$$\frac{\partial\,(ABC)}{\partial\,\mathrm{Vec}(B)}=A\,\mathrm{II}_BC$$

$$\frac{\partial\,(AB^TC)}{\partial\,\mathrm{Vec}(B)}=A\,\mathrm{II}_B^TC$$

$$\frac{\partial\,(BCB^T)}{\partial\,\mathrm{Vec}(B)}=\mathrm{II}_BCB^T+BC\mathrm{II}_B^T$$

Example: The standard Kalman filter gain matrix is derived by[2]

$$K=PH^TR^{-1}$$

The sensitivity of the gain matrix K with respect to the measurement distribution matrix H is simply

$$\frac{\partial K}{\partial\,\mathrm{Vec}(H)}=P\mathrm{II}_H^TR^{-1}$$

Symmetry Properties:

1) If A,B are symmetric matrices $\Rightarrow$ Vec $(A\,\mathrm{II}B)$ is symmetric.

2) If C,D are general square matrices $\Rightarrow$ Vec $(C\mathrm{II}^TD+D\mathrm{II}^TC)$ is symmetric.

3) If C,D are general square matrices $\Rightarrow$ Vec $(C\mathrm{II}D+C^T\mathrm{II}D^T)$ are symmetric.

All of the preceding properties can easily be verified by expanding the matrices.

Conclusion

Using the previously defined Vec operator and the expanded identity matrix notation, the derivative of a complex matrix equation with respect to an array can be easily derived in a manner similar to the normal differentiation procedure of scalar equations. The concise notations are particularly suitable for digital computer implementation of complicated matrix equations.

References

[1]Graham, A., *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood Limited, Chichester, England, UK, 1981.

[2]Bryson, A. E., and Ho, Y. C., *Applied Optimal Control,* Hemisphere, Washington, DC, 1975.